Graphic Organizers Applied to Higher-Level Secondary Mathematics

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Abstract. A review of popular approaches to teaching mathematics that emphasize nonverbal skills, such as using manipulatives or problem-solving schema, shows that they are often not immediately applicable to some important areas of secondary algebra. However, graphic organizers in various forms have been widely suggested and researched as an intervention approach to improve reading comprehension. In this article, suggestions for modifying graphic organizers to make them more applicable to teaching higher-level mathematics concepts and procedures are proposed. Using an appropriately modified graphic organizer to teach higher-level mathematics skills may help students with relatively weak verbal skills and strong nonverbal reasoning skills to be more successful in mathematics. This subgroup of students with learning disabilities has been consistently identified in several schemes for subtyping mathematical disorders developed from empirical evidence. Excerpts of two sample scripted lessons based on this approach, one for the solving of systems of linear equations in three variables and one for the concept of negative integer exponents, are included.

Yesterday, the first author received an e-mail from a former student. The note reminded us why we wanted to write this article. It wasn't the content of the note as much as the student himself that was important. For six years the first author taught secondary mathematics courses at a small private school dedicated to students with learning disabilities; this student had been in several of his classes. As a high school senior the student took an introduction to differential calculus course with

the first author and had also demonstrated some talent as an artist, particularly in oil painting. At the same time, he was reading at about a fifth-grade level despite several years of specialized reading instruction, and he found memorization of basic mathematics facts nearly impossible. This student was one of many students who seemed capable of understanding the complex content of advanced secondary mathematics in spite of substantial reading and language deficits.

While traditional classroom instruction relies heavily on reading and language-based instruction, my challenge with these students was finding ways to give these students access to the information without relying so heavily on these reading and language skills. The point here is that students come to class with a wide range of relative strengths and weaknesses in cognitive abilities as well as in achievement. This is particularly true of students with learning disabilities. It becomes especially important for us, as teachers, to understand and respond to the relationships between these differences and various approaches to instruction. However, instruction at higher levels of secondary mathematics is particularly dependent on language skills, suggesting that students with weak verbal abilities and strong nonverbal abilities are being underserved in this area.

The purposes of this article are to provide a rationale for applying graphic organizers to instruction in secondary mathematics, particularly algebra, and to make recommendations for the creation and application of graphic organizers for this material. More specifically, the article provides a rationale for using graphic organizers based on the fact that students with language disorders often struggle with conventional mathematics instruction that places large demands on language skills. Some current practices in mathematics instruction reduce this emphasis on language skills, but are not readily applicable to some higher-level mathematics skills. Following this discussion, review of the use of graphic organizers to improve reading comprehension

is used to identify features of graphic organizers that contribute to their effectiveness. These important features of graphic organizers are then applied to mathematics content. Two specific examples of direct instruction in secondary algebra using graphic organizers are described. Finally, some general recommendations for both teachers and researchers conclude the article.

RATIONALE FOR THE USE OF GRAPHIC ORGANIZERS

Language skills and mathematics skills are not mutually exclusive. Mathematics instruction often assumes that the learner has typical language skills. However, this is not necessarily the case for students with learning disorders. Students with language disorders can be at a disadvantage when mathematics instruction relies heavily on language skills. Some approaches to mathematics instruction, such as the use of manipulatives and graphs, offer these students an alternative to dependence on receptive language skills. However, these approaches have generally been applied only to basic mathematics skills, and they may not be applicable to some more advanced mathematics skills. The literature on reading comprehension has shown that a variety of graphic organizers have been effective in improving student performance. This section ends with some suggestions for modifying the definition and use of these graphic organizers so that they may be applied to mathematics instruction.

Relationships Between Language Skills and Arithmetic Skills Deficits

Mainstream classroom instruction in mathematics assumes adequate language and reading competence (Bley & Thornton, 1995; Moses & Cobb, 2001; Rivera, 1998). This reliance presents a challenge for students who demonstrate learning disorders. Students who have difficulties with language can struggle with mathematics instruction in a variety of ways (Bley & Thornton, 1995; Miller & Mercer, 1998). For example, they may have difficulty following or understanding instructions, understanding mathematical terms, reading or understanding word problems, recognizing variations on tasks, verbalizing what they know, or identifying irrelevant information. These students are found in classrooms along with other students who do not have to face these language challenges.

Over the years, teachers and researchers have developed several techniques for illustrating mathematics concepts that are not dependent solely on reading and language comprehension (Harris, Miller, & Mercer, 1995; Jitendra & Hoff, 1996). These techniques include the use of concrete objects and spatial representations of relationships that may not rely as heavily on language and reading skills as conventional instruction does. Little empirical evidence for aptitude-treatment interactions yet exists to support the claim that these approaches would benefit students with language disorders more than approaches that are more dependent on language skills. However, this approach is consistent with a long tradition in special education of modifying instruction to take advantage of relative cognitive strengths. Further, efforts to subtype developmental arithmetic disabilities based on empirical evidence are at least consistent with this view.

Several authors have proposed subtypes of developmental arithmetic disabilities that are derived from empirical evidence (Geary, 1993, 2000; Kosc, 1974; Rourke, 1989, 1993; Rourke & Conway, 1998). A tentative comparison of these approaches is outlined in Figure 1. The figure indicates ways subtypes from one approach might be comparable to, subsumed by, or overlap with subtypes from other approaches. Entries in the figure indicate the name assigned to each subtype and indicate areas of relative weakness for each of the subtypes. A brief review of each of these subtyping approaches follows.

Rourke has suggested that students with specific learning disabilities (LD) may be categorized into two general groups and, further, that the arithmetic difficulties of each group may have different origins (Rourke, 1989; Rourke & Conway, 1998). Rourke's Group R-S was substantially deficient in reading and spelling but not as weak in arithmetic performance (although not performing up to age norms in this area). Rourke suggested that their weakness in arithmetic might be secondary to their reading problems. This group was described as having verbal learning disabilities (VLD).

Rourke	Kosc	Geary
Group R-S	**Verbal**	**Semantic Memory**
Reading	Naming numbers	Fact retrieval
Spelling	**Lexical**	High error rates
(also poor Arithmetic)		Associated errors
	Reading numbers	
	Graphical	
	Writing numbers	
Group A	**Practognostic**	**Visuospatial**
Arithmetic	Concrete and pictorial	Misalignment and
Visual/spatial	manipulations	misperception of numbers
Tactile		Geometry
Abstract thinking	**Ideognostical**	**Procedural**
	Mathematics concepts	Immature procedures
	Operational	Sequencing
		Conceptual delay
	Choosing and carrying out	
	operations	

FIGURE 1   Relative weaknesses in mathematics based on subtypes of developmental mathematics disabilities.

Rourke's other specific group, Group A, was performing at normal levels in reading and spelling, but was impaired in arithmetic performance. This group also typically performed poorly on nonverbal reasoning and concept-formation tasks, as well as spatial and tactile tasks. As a result, Rourke further described them as having nonverbal learning disabilities (NLD) (Rourke, 1989; Rourke & Conway, 1998). This NLD group showed substantial difficulty in mathematics without evidence of language or reading problems.

Geary has proposed three subtypes of developmental mathematical disorders. He attributed one subtype to deficits in semantic memory and noted that it "appears to occur with phonetic forms of reading disorder" (2000, p. 6). Thus, Geary's semantic memory subtype of mathematical disorders is linked to reading problems. This semantic memory group had particular difficulty with mathematics fact retrieval. Geary's second subtype of mathematical disorders involves problems with procedural skills. Students with this subtype are likely to use immature procedures such as counting on fingers. They are also likely to produce frequent errors in carrying out procedures, may have developmental delay in understanding concepts, and have difficulty with sequencing multistep procedures. The third subtype in Geary's scheme reflects visuospatial deficits and is evidenced by misalignment of numerals, misinterpretation of place value, and difficulties with geometry. Geary reports that there are no clear relationships between these latter two subtypes of mathematical disorders and reading. Thus, Geary has proposed one subtype of arithmetic disability related to language or reading problems, and two that are not clearly related to language or reading.

Kosc (1974; Sharma, 1986) recommended a more detailed model of mathematical disabilities than the others described here. In his model, developmental dyscalculia is classified into six different forms that may manifest separately or in combination. Kosc's system suggests three subtypes of dyscalculia that are at least analogous to written or oral language skills, and three that are not clearly related to language or reading skills. Kosc's verbal dyscalculia is evidenced by deficits in the ability to verbally name mathematical terms and relations. The person's difficulties with labeling or word finding could interfere with reading particularly, although his or her nonverbal reasoning skills may be intact. Practognostic dyscalculia is a disability that interferes with mathematical manipulation of concrete and pictorial representations. Those with practognostic dyscalculia may have difficulty counting physical objects or comparing the relative sizes of two similar objects. Lexical dyscalculia refers to difficulties with reading mathematical symbols. This form of developmental dyscalculia often occurs in combination with reading difficulties (Sharma, 1986). Graphical dyscalculia is a deficit in manipulating mathematical symbols in writing. In some cases students may not be able to write the symbol for a dictated number, although they can write the word(s) for that number. Alternatively, digits for a dictated number may be written correctly but out of order. In severe cases students may not even be able to copy numbers accurately. This is an arithmetic analogue to a written language difficulty and Sharma (1986) has noted that graphic dyscalculia (as well as lexical dyscalculia) typically occurs with reading and written language disorders. Ideognostical dyscalculia refers to a disability in understanding mathematical concepts and calculations. Tasks designed to test for this deficit may be performed less efficiently by using verbal skills, but essentially these are nonverbal skills. Operational dyscalculia refers to a disturbance in the ability to carry out mathematical operations. As with ideognostical dyscalculia, students with operational dyscalculia are struggling with nonverbal skills, although these tasks could conceivably be performed through less efficient verbal means.

The striking feature of Figure 1 is that all these subtyping schemes share one common criterion for distinguishing between subtypes. Each one of these approaches considers one or more subtypes that are related to verbal disabilities, as well as one or more subtypes that are related to nonverbal disabilities. Students with specific verbal disabilities may benefit from interventions that emphasize nonverbal abilities, while those students with specific nonverbal disabilities may benefit more from interventions that take advantage of their stronger verbal abilities. Although the relationships between specific mathematics disability subtypes and specific interventions has not been thoroughly investigated, it is important to look at how this might be done. However, it must be borne in mind that such an approach is merely conjecture at this point.

Nonverbal Approaches to Mathematics Instruction

As indicated earlier, little work has been done to show that different interventions are effective for different mathematics disability subtypes. This research void is consistent with the widely recognized fact that mathematics disabilities have been the focus of far less research than other areas of learning disabilities, particularly reading (Geary, 2000; Padget, 1998; Rourke & Conway, 1998). By contrast, in the area of reading disabilities, researchers have found evidence of interactions between different interventions and different subtypes of dyslexia (Bakker, Bouma, & Gardien, 1990; Fiedorowicz & Trites, 1991; Robertson, 2000). Their findings certainly don't provide clear evidence for analogous interactions with respect to mathematics disabilities. However, the findings in reading disabilities do lend credence to an expectation of similar interactions in mathematics disabilities and suggest an area that should be investigated. Investigators have certainly recommended different intervention approaches for teaching mathematics to students with different learning abilities (Baroody, 1996; DeLuca, Rourke, & Del Otto, 1991).

It follows from the mathematics disability subtyping patterns that students with dyscalculia related to language and reading deficits would benefit more from instructional approaches that rely on using visual/spatial and visual/perceptual skills. For example, using the popular concrete-semiconcrete-abstract (CSA) sequence and manipulatives to teach mathematics facts would take advantage of visual/spatial skills (Harris, Miller, & Mercer, 1995; Marsh & Cooke, 1996; Miller & Mercer, 1993). On the other hand, students with nonverbal types of dyscalculia should benefit more from verbally-based approaches. Rote memorization, direct instruction, constant time delay procedures, strategy instruction, and "count-by" techniques rely more exclusively on verbal abilities and should more directly benefit students with stronger verbal skills (Koscinski & Gast, 1993; Mattingly & Bott, 1990; McIntyre, Test, Cooke, & Beattie, 1991; Swanson, Hoskyn, & Lee, 1999). Although empirical evidence for all these techniques has shown them to be generally effective, these studies have not distinguished between participants on the basis of the relative strengths of their verbal and spatial skills. In other words, aptitude-treatment interactions have not been investigated.

The use of manipulatives and the CSA approach to mathematics instruction is widely advocated for basic skills instruction. Miller and Mercer (1993) identified materials for CSA instruction in basic operations and place value, money, time, weight and measurement, fractions, decimals, percents, and geometry. Quite a few papers support the use of manipulatives in general, and the CSA approach in particular, for instruction in these areas of mathematics for students with and without identified disabilities (Cain-Caston, 1996; Fueyo & Bushell, 1998; Maccini & Hughes, 2000; Stellingwerf & Van Lieshout, 1999).

Nevertheless, there are some important theoretical and practical limitations to the use of manipulatives in mathematics instruction. More than 50 years ago, Weaver (1950) pointed out that to be effective, manipulatives must be carefully matched with the abstract understanding that is the goal of the instruction. This is important because "concrete materials do not automatically carry mathematical meaning for students" (Thompson, 1994, p. 557). Teachers must not only make sure that the materials are closely related to the abstract content (Bright, 1986; Hynes, 1986; Miller & Mercer, 1993), but that the instruction itself is designed to make the relationships between the materials and the concepts clear (Miller & Mercer, 1993; Uttal & DeLoache, 1997).

A second concern about the use of manipulatives and CSA teaching approaches is that they are typically applied to younger students and basic skills. Kennedy and Backman (1993) reported that there is support for the use of manipulatives with elementary and middle school children, but very little evidence for their use or effectiveness with secondary students. Maccini and Hughes (2000) found no studies validating the use of CSA to teach algebra except for relational word problems, and

further lamented the overall dearth of research on algebra instruction for students with learning disabilities.

A third limitation of the applicability of manipulatives and CSA teaching approaches is that they presume a concrete analog for the concept being taught. Secondary mathematics often includes some concepts that do not readily lend themselves to concrete representation. Consider the sine function in trigonometry. The sine function is often defined first by selecting one of the nonright angles in a right triangle, and then defining the sine function as the ratio of the length of the opposite side to the length of the adjacent side of that angle, excluding the hypotenuse. Although it is easy to create a concrete representation of a right triangle, it is another matter to devise a concrete representation of the ratio that defines the sine function itself. Certainly, we can move to the representational level. For example, a unit circle could be used. Nevertheless, the unit circle doesn't actually display a representation of the ratio that makes up the sine function either, even if it provides a means of displaying and comparing an infinite variety of triangles. A more specific visual representation of the sine function could come from a conventional graph of the function. However, this sine wave is also not clearly connected to the ratio of sides that defines the function. Thus, neither the unit circle nor the graph actually demonstrates the concept of the sine function. The same can be said of many higher-level mathematics concepts. Relevant concrete manipulatives are hard to imagine, and visual representations do not actually convey the meaning of the concept.

Moving from basic arithmetic skills to problem-solving and mathematics applications, visual/spatial approaches are largely absent from the intervention literature. Strategies using verbal mnemonic devices for remembering the steps, verbalization of student planning, scripted direct instruction, and group problem-solving activities all rely heavily on verbal abilities (Hutchinson, 1993; Montague, 1992; Montague, Applegate, & Marquard, 1993; Naglieri & Gottling, 1997; Wood, Rosenberg, & Carran, 1993). Mathematics texts typically encourage diagramming as one technique for problem solving, and this approach should favor students with stronger visual skills. However, far less empirical work is focused in this direction (Grossen & Carnine, 1990; Zawaiza & Gerber, 1993). Hegarty and Kozhevnikov found that "research to date has not demonstrated a clear relationship between use of visual-spatial representations and success in mathematical problem-solving" (1999, p. 684).

One exception to this scarcity of nonverbal approaches for problem solving derives from schema-based efforts to improve problem-solving skills. Jitendra and colleagues (Jitendra et al., 1998; Jitendra & Hoff, 1996; Jitendra, Hoff, & Beck, 1999) have used a schema-based approach to improve word problem-solving abilities in students who are at risk and students who have learning disabilities. Generally, students were trained to distinguish between three different types of word problems, and then were taught specific diagrams,

provided by the investigators, to solve each type of problem. Each diagram consisted of a few geometric shapes and a few words. The diagrams were designed to indicate specific places for entering data from the problem, and thereby indicate the relationships between those data, which would in turn indicate what actions to take to solve the problem. This work is quite similar to earlier work (Fuson & Willis, 1989; Willis & Fuson, 1988) with second-grade students in general education classes who were learning to solve one-step addition and subtraction problems. This approach is based on the theory that successful problem solving derives from the effective selection and application of schemata that represent the elements of the problem in a meaningful way, and that these schemata can be taught directly.

As students move into higher levels of mathematics, spatial/visual teaching techniques are not only less well documented, they are also more difficult to devise. The concrete-semiconcrete-abstract approach to mathematics instruction is readily applied to elementary-level mathematics skills such as basic operations, use of money, and place value. Similarly, spatial/visual techniques and manipulatives for teaching skills in fractions, decimals, exponents, basic algebra, and Euclidean geometry are common. Mastery of these skills is adequate for students who are aiming for a high school diploma, but this level of skill attainment falls short of the needs of students hoping to go on to traditional four-year post-secondary programs. These students will need skills that are covered in a second course of algebra as well as in functional analysis and calculus courses. There is an intervention vacuum in spatial/visual techniques for working at this more advanced level of mathematics with students with LD whose nonverbal reasoning skills are strong but whose verbal skills are weak. These students have the reasoning skills to handle the concepts but lack the verbal skills to gain access to them. Virtually no literature addresses this concern. Further, college-level texts for preservice teachers that survey intervention approaches for students with learning disabilities and other mild disabilities fail to even mention these higher-level mathematics skills, much less make recommendations for their instruction (Bender, 1996; Bley & Thornton, 1995; Jones, Wilson, & Bhojwani, 1998; Mastropieri & Scruggs, 1994; Meese, 1994).

Graphic Organizers in Reading Comprehension

One possible source of ideas regarding the use of graphic organizers to support higher-level skills is the reading skills literature. Workers in this area have been looking at an approach to improving comprehension using visual displays to represent relationships between the pieces of information in a text. Perhaps these graphic organizers can be borrowed and applied to teaching higher-level mathematics concepts.

The evolution and effectiveness of graphic organizers for reading comprehension has been reviewed several times (Alvermann & Swafford, 1989; Dunston, 1992; Moore & Readance, 1984; Rice, 1994; Robinson, 1998; Swafford & Alvermann, 1989) and will only be briefly summarized here. Ausubel (1960) recommended the use of advanced organizers as a means of improving comprehension of classroom learning tasks. Advanced organizers were short prose passages intended to provide scaffolding based on prior knowledge to facilitate incorporating new knowledge. Subsequent investigators suggested modifying the advanced organizer by using key vocabulary and short phrases rather than prose, and arranging these verbal elements in a visual/spatial configuration that would represent the relationships between the verbal elements (Earle, 1969; Estes, Mills, & Barron, 1969). These displays became known as *structured overviews*. However, research failed to consistently support the effectiveness of either advanced organizers or structured overviews, at least as they were applied to reading comprehension.

Arguing that the problem lay in student engagement rather than with the theory, subsequent researchers have attempted to show that graphic organizers can improve reading comprehension when the students are actively engaged in working with or creating them. One approach has been to give students incomplete graphic organizers prior to reading. Students were required to complete the graphic organizer as they read. Alvermann and others have demonstrated fairly consistent positive effects of graphic organizers using this approach to engage the students (Alvermann & Boothby, 1986; Alvermann, Boothby, & Wolfe, 1984; Barron & Schwartz, 1984; Boothby & Alvermann, 1984). A second approach is to train students to create their own graphic organizers (in this case called graphic post-organizers), then give them a novel text to read, and assess their comprehension (Bean, Singer, Sorter, & Frazee, 1986; Dunston & Ridgeway, 1990; Griffin & Tulbert, 1995). Studies have demonstrated even more consistently positive results for post-organizers as opposed to preorganizers (Dunston, 1992; Griffin & Tulbert, 1995; Rice, 1994).

One criticism of these studies relates to method. Studies to date have often not been specific about the depth and type of instruction that students received (Dunston, 1992; Robinson, 1998), making evaluation and replication difficult. It is often not clear whether or how much detailed instruction students have received concerning the relationships among verbal elements in organizers, how to use the organizers to learn the information, what information the graphic elements of the organizer represent, and how to construct organizers of their own. This kind of information is critical to an understanding of what makes graphic organizers effective.

A second concern is that these studies typically assess vocabulary and factual units as their dependent variables, rather than relationships (Robinson, 1998). In spite of the fact that graphic organizers are intended to convey relationships visually, learning of this relational

information is often not evaluated. A revealing exception to this pattern is a study by Kiewra using researcher-created organizers with 44 college students (Kiewra, Dubois, Christian, & McShane, 1988). Kiewra compared the effectiveness of an outline, which is one dimensional, to a matrix organizer, which is two dimensional. Both organizers had identical content. Kiewra found that students using the matrix organizer were significantly more able to remember the relationships highlighted by the horizontal dimension of the matrix, which was not available to the students using the outline (Kiewra et al., 1988).

Adapting Graphic Organizers to Mathematics Instruction

Given that all the mathematical disorder subtyping schemes described earlier have identified a group of students with relatively weak language skills and relatively strong spatial and nonverbal reasoning skills, instructional approaches in mathematics that take advantage of the strengths of these students may improve their academic progress in mathematics. Graphic organizers rely on visual/spatial reasoning skills more so than conventional teaching approaches and may be applied to the teaching of higher-level mathematics. To this end, three important modifications to graphic organizers such as those used for reading comprehension are recommended here. First, the content of the graphic organizers in mathematics would no longer be verbal elements—words, phrases, and sentences. Rather, the content would be mathematical analogues to these verbal elements—numbers and other symbols, expressions, and equations.

Second, it is important to keep in mind that while acquiring basic mathematics skills often involves learning facts, higher-level skills are concerned with concepts, patterns, and processes. As such, the goal in using graphic organizers for higher-level mathematics is not to learn the mathematical elements. There is no point in memorizing numbers, expressions, and equations. The goal is to recognize and learn the patterns that connect these elements. This means that the graphic display, the spatial arrangement of the mathematical elements, must support the information to be learned.

Third, graphic organizers would be an integral part of good instruction, not a substitute for instruction. Graphic organizers would be incorporated into lessons such that the relationships students are to learn are explicitly taught and connected to the graphic organizers. Direct strategy instruction has been shown to be effective in helping students with learning disabilities learn and generalize strategies across a variety of subjects (Deshler, Ellis, & Lenz, 1996; Deshler & Lenz, 1989; Deshler & Schumaker, 1993), as well as mathematics in particular (Mercer & Miller, 1992). Further, in a meta-analysis of intervention studies for students with learning disabilities, approaches that include direct in-

struction and/or strategy instruction generally produced greater effect sizes than approaches without these features (Swanson et al., 1999). With respect to graphic organizers applied to mathematics, this may include not only instruction in the visual elements of the display, but also instruction in the use of the organizer as a learning guide, and the construction of organizers, when appropriate for the instructional objectives.

When designing and using a graphic organizer for instruction at any level, it is important to remember that the purpose of the organizer is to reinforce relationships among the elements of the organizer, rather than the elements themselves. This point leads to three important responsibilities for teachers when designing instruction using graphic organizers. First, just as with the selection of manipulatives for mathematics instruction, teachers must design graphic organizers that are clearly related to the relationships being taught. Second, as was emphasized for the use of both manipulatives in mathematics and graphic organizers in reading comprehension, the important relationships and concepts must also be directly connected to the manipulatives or the organizer during instruction: simply presenting the organizer without connecting it to the relationships is not enough. Third, presumably, teachers would choose to use graphic organizers because they believe that it is important for their students to understand the relationships being represented. If so, then teachers need to review their assessment to ensure that they are actually looking at students' conceptual understanding of relationships instead of, or in addition to, their abilities at carrying out procedures and recalling facts.

Winn's (1991) discussion of maps and diagrams provides useful guidelines for creating graphic organizers. Winn pointed out that relationships between elements of a diagram may be indicated by their relative positions to other elements in the diagram, and also by their relative positions to the frame within which the diagram is placed. For example, the conventional outline format for notes demonstrates relationships between items based on how far they are indented relative to each other and whether they are above or below each other. In contrast, a typical city map has numbered and lettered coordinates in the margins so that a user can find the area of a street or point of interest on the map relative to the margins. In this case, the relationships of elements in the map to the frame of the map convey information in addition to what might be learned by looking at the map without this frame. These two types of relationship suggest that if we wish to emphasize the relationships between elements within a graphic organizer, our instruction should focus on the connections between the relative positions of the elements and their relationships to each other. On the other hand, if we wish to emphasize the relationships between elements within a graphic organizer and other patterning information, we should design a graphic organizer such that the frame indicates that patterning information, and then focus instruction on the connections between the relative positions of the elements and the frame of the organizer.

TWO APPLICATIONS OF GRAPHIC ORGANIZER PRINCIPLES TO HIGHER-LEVEL MATHEMATICS

To illustrate these points, excerpts from two sample lessons using graphic organizers to support teaching algebra concepts and procedures are offered here. In the first example, a graphic organizer with no frame is used to demonstrate relationships between negative and positive integer exponents. The second example uses a graphic organizer with a frame to guide the process of solving systems of three linear equations in three variables, and also to reinforce some of the concepts that justify the steps in that process.

The Concept of Negative Integer Exponents

Negative integer exponents are typically introduced to secondary students in a first course in algebra (Collins et al., 2001; Larson, Boswell, Kanold, & Stiff, 2001). However, the topic is often presented simply as a definition, in the form of an algebraic equation, that must be memorized, rather than as a meaningful concept that is related to information the students have already covered. In an extreme case, one text announced that "negative exponents cannot be 'understood' because they are the result of a definition, and thus there is nothing to understand" (Saxon, 1997, p. 29). This is, of course, an absurd statement mathematically. The definition of negative exponents is not an arbitrary whim. It is a perfectly logical extension of what students already have covered concerning positive exponents. This relationship between students' understanding of positive integer exponents and concepts behind negative integer exponents can be pointed out with the help of a graphic organizer. It follows that instruction using a graphic organizer for this material should stress relationships between elements in the organizer, rather than relationships between elements of the organizer and a frame. In fact, the organizer suggested here has no frame at all.

The lesson may be introduced by letting students know that they will be learning about negative exponents and then giving them an opportunity to brainstorm and share ideas related to exponents as they already understand them. Usually in the process of this exchange someone will suggest a basic exponent mathematics fact, such as $2^3 = 8$. If not, such an example can be solicited more directly, as in this brief sample dialogue where teacher questions and statements are indicated by Roman type, and typical student responses are indicated by *italics*.

What does 2^3 equal?
8.
What does 2^4 equal?
16.
What does 2^5 equal?
32.

$$2^6 = 64$$

$$2^5 = 32$$

$$2^4 = 16$$

$$2^3 = 8$$

$$2^2 = 4$$

$$2^1 = 2$$

FIGURE 2   Left column of a graphic organizer for teaching negative integer exponents.

Using this kind of exchange, a column of exponent mathematics facts is built as suggested in Figure 2. Next the students are given the opportunity to recognize patterns in this information.

When we go up from 2^3 to 2^4, how did the exponent change?
It went up by one.
How did it go up? Was that adding, subtracting, multiplying, or dividing?
Adding.
When we go up from 2^4 to 2^5, how did the exponent change?
Plus one. Added one.
If we went up one more row, what would the next exponent be?
6.
How do you know?
You just add one.
Good. Now let's look at the values on the other side of the equal sign. Can we get from 8 to 16 by adding 1?
No.
How can we get from 8 to 16?
You double it. Times two.
Does that still work when we go up from 16 to 32?
Yes.
Using that pattern, what will 2^6 equal?
64.
Check that on your calculators. Is that right?
Yes.

Once this pattern of incrementing exponents and multiplying values is established, the next step is to reverse directions and look at the same relationships.

If we keep going will we eventually reach negative exponents?
No.
Why not?
They keep getting bigger.

Well then, let's look at what we're doing and try to figure out how to go backward. If we go down from 2^6 to 2^5, how does the exponent change?
One smaller. Subtract one.
If we go down from 2^5 to 2^4, how does the exponent change?
One smaller. Subtract one.
If we go down from 2^4 to 2^3, how does the exponent change?
One smaller. Subtract one.
Is that the same rule as when we went up?
No. We added.
If we keep going down, what will the next exponent be after 3?
2.
Why?
We go down one.
Okay, good. Now let's look at the values on the other side of the equal sign again. When we went up we multiplied by 2. What happens when we go down from 64 to 32?
Divide by 2. It's half.
What happens when we go down from 32 to 16?
Divide by 2. It's half.
What about 16 to 8?
Divide by 2. It's half.
If we use the same rule, what is the next value down?
4.
Does that work? Does 2^2 really equal 4?
Yes.
Great. Let's keep going. The exponents go down 5, 4, 3, 2. What would the next one be?
1.
How do you know?
You go down one.
What about the values? They go 32, 16, 8, 4. What would the next one be?
2.
Why?
You divide by 2. You halve it.
Is that equation right? Does 2^1 equal 2?
Yes.
Are we getting closer to negative exponents now?
Yes.

Up to this point, the column of exponent facts has grown vertically, first upward and then downward. Because one intent of the graphic organizer is to draw attention to the relationship of positive integer exponents to negative integer exponents of the same value (as well as the uniqueness of zero exponents in that respect), the relative position of the exponent mathematics facts changes as the lesson proceeds from here. Figure 3 shows a completed display for these facts.

Let's do another row. What is the next exponent? 5, 4, 3, 2, 1, . . .?
0.
Good. Now, what's the next value? 32, 16, 8, 4, 2, . . .?
1.

$$2^6 = 64 \qquad 2^{-6} = \tfrac{1}{64}$$

$$2^5 = 32 \qquad 2^{-5} = \tfrac{1}{32}$$

$$2^4 = 16 \qquad 2^{-4} = \tfrac{1}{16}$$

$$2^3 = 8 \qquad 2^{-3} = \tfrac{1}{8}$$

$$2^2 = 4 \qquad 2^{-2} = \tfrac{1}{4}$$

$$2^1 = 2 \qquad 2^{-1} = \tfrac{1}{2}$$

$$2^0 = 1$$

FIGURE 3   Completed graphic organizer for teaching negative integer exponents.

How do you know?
Divide by 2.
Have we reached negative exponents yet?
No.
How much further do we have to go?
One more.
OK, let's do the next one. What's the next exponent?
−1.
Good. Now what's the rule for finding the next value?
Divide by 2.
OK, what's 1 divided by 2? Give me the answer as a fraction.
One half.

The lesson can proceed in an obvious way, reiterating these patterns until the right-hand column of the organizer is completed. At this point the students are encouraged to find patterns in the organizer. Students rarely have difficulty recognizing that the exponents of the left column are opposites of those in the right column while the values in the left column are reciprocals of those in the right column. If they do have difficulty, horizontal lines can be drawn through the array to make the relationship clearer. This may be particularly helpful for students who have visual/spatial perception problems.

The relationship between the entries in the left versus the right columns is equivalent to the algebraic definition of negative exponents typically given in textbooks. Further, students can be led to an appreciation of negative integer exponents as serial division by the base number in the same sense that positive integer exponents can be thought of as serial multiplication by the base number. The first author has applied this approach for several years when working with students with learning and attention problems. During those classes it was common for students to prompt each other with

comments like, "Don't you remember the ones that were side-by-side?"

An added bonus for this approach accrues from presenting two of these organizers, with different bases, next to each other. In each case the exponential mathematics fact with the zero power is isolated at the bottom of the organizer. Students can be led to note that when the exponent is zero, the value of the exponential expression is always one. Of course, the fact that zero to the zero power is undefined and an exception to the pattern should be made clear.

Here a simple graphic organizer is used to emphasize meaningful concepts and patterns relating to negative integer exponents. Following Winn's (1991) distinction between two types of relationships in maps and diagrams, this graphic organizer relies on the relative positions of elements within the organizer to indicate these concepts. No frame was necessary. In the next example, a multistep procedure is being taught with a goal of helping students understand the concepts that justify those steps. In this case a meaningful frame is appropriate.

Solving Systems of Three Linear Equations with Three Variables

Excerpts of a lesson tackling the solving of systems of linear equations in three variables are presented here. This task involves many steps, and also requires some fairly astute decision making along the way. Although the systems can be solved by rote procedure (as is typically done in computer programs), the ability to look ahead can often greatly simplify the process. The many steps involved and the decision making that is required can make this a challenging task for students, especially students with learning problems. The procedure is the center of the focus for the lesson, with relevant mathematical concepts tied to particular steps to make them more meaningful. To help students keep these broader concepts in mind, a graphic organizer has been designed in which the relative positions of the equations of the problem to the frame of the organizer are emphasized.

The lesson excerpted here is intended to show how a graphic organizer can be used to help students track and make sense of the processes involved in solving systems of linear equations by linear combination. The relationships between elements of the graphic organizer and these goals is described in more detail below. The graphic organizer was not designed to support a particular view of the purposes of linear systems of equations or a justification for solving them. Certainly, these are important concerns that teachers should also address through other elements of their lessons.

The specific system solved in this example was constructed to be uncomplicated. Thus the example does not specifically address variations such as systems where one or more of the equations have fewer than three variables, or systems in which initial coefficients of variables do not cancel when added without

multiplying the equations by constants first. These complications will be anticipated and discussed within, as well as after, the lesson.

Prior Knowledge

Prior knowledge elements necessary for this lesson include: (1) combining (adding) linear equations, (2) substituting values in place of variables in equations, (3) solving linear equations in one variable, and (4) recognizing that linear equations in one variable typically have unique solutions (i.e., they can be "solved"), whereas linear equations with more than one variable do not. Additional prior knowledge required for more challenging examples could include: (1) the fact that multiplying linear equations by constants does not change their solutions, (2) the ability to either find least common multiples, or use the fact that multiplying two numbers by each other yields the same product (commutative property of multiplication), and (3) the fact that multiplying an equation by negative one (-1) reverses the signs of each term in the equation.

The Graphic Organizer

The frame for the organizer is shown in Figure 4. The lines comprising the borders of the rectangles serve to divide the symbolic content into cells based on meaningful distinctions. They also serve to emphasize the relative positions of various symbolic content elements to each other. In a typical system of equations, the solution of the problem using the graphic organizer involves working from cell to cell in a clockwise direction starting with the top left cell. This clockwise movement is intended to help students anticipate where they will be working next.

Another pattern in the solving process is illustrated by comparing the top row to the bottom row of the organizer. The top row is used to combine equations
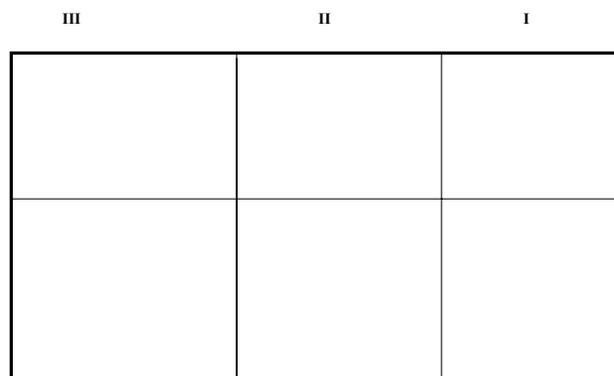

FIGURE 4 Blank graphic organizer for solving systems of linear equations in three variables.

in order to eliminate variables until an equation of one variable is produced. Once this equation is found, the bottom row serves to guide the successive solving for, and substituting of, values for variables until the entire system is solved.

Each column is headed by a Roman numeral, with the Roman numerals in descending order from left to right—III, II, I. While the Roman numerals are certainly symbolic elements, they are elements of the frame of the graphic organizer, rather than symbolic elements of the content of the problems to be solved using the graphic organizer. The left column is labeled "III." Equations containing three variables will be placed in the top cell of this column, while equations with two variables will be placed in the top cell of the middle column, and equations with only one variable will be placed in the top cell of the right column. Thus the Roman numerals in the frame indicate the number of variables in the equations below them, and the relative lateral position of equations to each other also indicates more or less variables when moving to the left or right, respectively. They also serve to indicate that one intermediate goal in solving systems of linear equations is to derive equations that contain fewer variables.

Specific values for each variable are solved for across the bottom of the graphic organizer. The first value is found in the bottom right cell, whose column is headed by the Roman numeral "I." The second value is found in the middle cell of the bottom row, and the third value is found in the left cell of the bottom row. Thus the Roman numeral headings coincide with the nth variable being solved for and the values are found in a lateral sequence from right to left. Again, both relative positioning of symbolic content elements to each other, and the position of these elements relative to the frame, indicate relationships between the elements.

The Lesson

An appropriate anticipatory set could be based on a contrived situation involving students in the classroom. One important goal of an anticipatory set is to spark some interest in students to solve the problem, and simultaneously provide a justification for learning to solve this type of problem. For example, Xavier, Yentl, and Zenobia (X, Y, and Z, respectively) each planned to deposit a few dollars in the classroom bank. However, they are considering changing these deposits, or even withdrawing money instead. We know that if X and Z deposit double their original amounts, and Y deposits four times her original amount, the bank balance will increase by $16. Similar descriptions can be constructed for two more equations involving deposits and withdrawals relative to the original plans of the three participants, and the resulting change in bank balance. The question to be answered is, "What were the original planned deposits of the three participants?"

Students can help construct a set of three linear equations in three variables based on the problem, such as $2x + 4y + 2z = 16$, $-2x - 3y + z = -5$, and $2x + 2y - 3z = -3$. The first step in the process is to establish a connection between the graphic elements of the organizer (boxes, columns, and Roman numerals) and the mathematical elements (equations). In the excerpts that follow, the teacher's questions and comments are in Roman type, likely student responses are in *italics*, and comments on the use of the graphic organizer are in parentheses.

What number is at the top of this column (indicating the right column)?
One.
Yes. Now, what number is at the top of the center column?
Two.
Good. What number is at the top of this column (indicating the left column)?
Three.
Take a look at the first equation. How many variables are in that equation?
Three.
If this is the Roman numeral for three, and the first equation has three variables in it, which column do you think we are going to write the first equation in?
The left. Under the three.

The first equation is entered as indicated in Figure 5. A connection has been introduced between the number of variables in an equation and the column in which that equation appears. This connection anticipates the complication of dealing with systems in which some equations may have fewer than three variables.

Now, how many variables are in this first equation?
Three.
What number is at the top of this column?
Three.
Coincidence?
I think NOT!

III	II	I
$2x + 4y + 2z = 16$		
$-2x - 3y + z = -5$		
$2x + 2y - 3z = -3$		

FIGURE 5   Graphic organizer for solving systems of linear equations in three variables as it may appear after the original equations have been entered.

As with most classrooms, these classes have established a ritual. Every time a pattern or connection is uncovered, the teacher cries out "Coincidence?" and the students chant back "I think NOT!" To reinforce the connection between the columns of the organizer and the variables in the equations, this process is essentially repeated for each of the other two equations, ending with the same ritualized response.

The next step is to combine equations in order to eliminate variables. As indicated earlier, this example is contrived such that variables will be eliminated in an orderly fashion without extra manipulations, because two equations can always be found such that the leading coefficients of one variable in both equations are already opposites. However, in preparation to deal with this complication in future lessons, this lesson includes dialogue to help students anticipate the complication. In addition, the dialogue reinforces the need to eliminate variables until they arrive at an equation with only one variable, which can be solved.

> How many variables are in this first equation?
> *Three.*
> Can we solve it?
> *No.*
> Why not?
> *It has too many variables. We need only one variable.*
> *There are lots of answers.*
> Can we solve either of the other two equations?
> *No.*
> Would it help if we had fewer variables?
> *Yes.*
> Take a look at the x-terms in the first two equations. What would we get if we added 2x and −2x?
> *Nothing. Zero. They cancel.*
> If we add the first two equations together will we have any x's left?
> *No.*
> Let's do it. What do we get when we combine 2x and −2x?
> *Nothing. Zero. They cancel.*
> What do we get when we combine 4y and −3y?
> *One y. y.*
> And if we combine 2z and z?
> *Three z.*
> What about when we combine 16 and −5?
> *Eleven.*
> What is the new equation we have created?
> *$y + 3z = 11$.*
> Which column did I put that equation in?
> *The middle one. The second one.*
> What Roman numeral is at the top of that column?
> *Two.*
> How many variables are in the new equation?
> *Two.*
> Coincidence?
> *I think NOT!*
> Take a look back at the first box we used. What is the first term of the middle equation?
> *−2x.*

FIGURE 6   Graphic organizer for solving systems of linear equations in three variables as it may appear after two linear equations in two variables have been found.

> What is the first term of the bottom equation?
> *2x.*
> What would we get if we combine those two terms?
> *Nothing. Zero. They cancel.*
> Try combining those two equations like we did the first two and show me what you get.

Here is an opportunity fairly early in the lesson for students to get some guided practice. They can receive feedback for their efforts, begin to get weaned from the scaffolding of the lesson, and perhaps reinforce some sense of self-efficacy. Figure 6 indicates how the graphic organizer might look after this step in the lesson.

Once all students have confirmed that they have $-y - 2z = -8$ for this equation, the group lesson proceeds. The connection between having two variables in the equation and placing the equation in the second column is reiterated. The fact that these two-variable equations cannot be solved is also acknowledged as previously done with the three-variable equations. Then the process of combining equations is repeated again for these two two-variable equations. Notice that the y variable is eliminated at this step in the example, but the lesson still goes through a confirmation of this result before combining the equations. This confirmation step allows room in the lesson to deal with future examples that will require multiplication of equations by constants before combining equations to eliminate variables.

When the two two-variable equations are combined in this example, they yield the equation $z = 3$. Thus it is not necessary in this example to perform any additional steps to solve a one-variable equation. However, as examples become more complicated, the solving of these equations would take place in the upper right-hand box. Students may need appropriate prompting and cueing to solve these equations. However, such guidance is not discussed here because the solution of linear systems in three variables is usually a topic for a second course in algebra, at which point solving these equations should be a well-learned element of prior knowledge. As with the two-variable equation, the connection between the column heading and the fact that the equation has only

III   II   I

$2x + 4y + 2z = 16$

$-2x - 3y + z = -5$

$2x + 2y - 3z = -3$

$y + 3z = 11$

$-y - 2z = -8$

$z = 3$

$z = 3$

FIGURE 7   Graphic organizer for solving systems of linear equations in three variables as it may appear after the first variable has been solved.

one variable is made explicit. Once the unique solution for this first variable has been determined, the solution is copied into the bottom right-hand box as directed by the arrow in Figure 7. Next, a connection is made between the organizer and the number of solutions that have been found.

> How many variables have we solved for so far?
> *One.*
> What number is at the top of the column above the solution?
> *One.*
> Coincidence?
> *I think NOT.*

The thought process for determining the next step in solving this system can be confusing for many students. It has been helpful in practice to tie the substitution steps to the knowledge that only a linear equation of one variable can typically be solved for a unique solution.

> How many variables are in the very first equation in column one?
> *Three.*
> If I substitute the "3" for the "z" in the very first equation in column one, how many variables will be left in the equation?
> *Two.*
> Will we be able to solve that equation if it has two variables in it?
> *No.*
> How many variables are in the second equation in column one?
> *Three.*
> If I substitute the "3" for the "z" in that second equation in column one, how many variables will be left in the equation?
> *Two.*
> Will we be able to solve that equation if it has two variables in it?
> *No.*
> How many variables are in the third equation in column one?

> *Three.*
> If I substitute the "3" for the "z" in the third equation in column one, how many variables will be left in the equation?
> *Two.*
> Will we be able to solve that equation if it has two variables in it?
> *No.*
> Will we be able to solve any equation in column one by substituting the "3" for the "z"?
> *No.*
> Take a look at the equations we put in the second column. How many variables are in the first equation in column two?
> *Two.*
> If I substitute the "3" for the "z" in that equation in column two, how many variables will be left in the equation?
> *One.*
> Will we be able to solve that equation if it has one variable in it?
> *Yes.*
> How many variables are in the second equation in column two?
> *Two.*
> If I substitute the "3" for the "z" in the second equation in column two, how many variables will be left in the equation?
> *One.*
> Will we be able to solve that equation if it has one variable in it?
> *Yes.*
> Which of those equations do you think will be easier to solve?
> *The first one.*
> Bring that equation straight down the column to the lower box, substitute the "3" for the "z" in the equation, and solve it.

Students typically choose the first of the two equations, although an identical solution results regardless of which equation is chosen and solved correctly. This is another opportunity for students to work more independently, get some individualized feedback, and in this case, practice a skill that has been previously acquired. Once the teacher has confirmed that all students have found $y = 2$, the same directed process can be repeated with appropriate substitutions to solve for $x = 1$. This yields the final solution to the problem: (1, 2, 3). Figure 8 demonstrates how the completed graphic organizer might appear.

Generalization

Students can be guided to generalize this approach in several ways. For example, systems of linear equations in three variables may not have all three variables in every equation. To solve these systems using the graphic organizer, it is helpful to place these equations with

III　　　　　**II**　　　　　**I**

$2x + 4y + 2z = 16$

$-2x - 3y + z = -5$ 　　$y + 3z = 11$

$2x + 2y - 3z = -3$ 　　$-y - 2z = -8$ 　　$z = 3$

$2x + 4(2) + 2(3) = 16$

$2x + 14 = 16$ 　　$y + 3(3) = 11$

$2x = 2$ 　　$y + 9 = 11$

$x = 1$ 　　$y = 2$ 　　$z = 3$

FIGURE 8   A completed graphic organizer for solving systems of linear equations in three variables.

fewer variables in upper boxes according to the number of variables that they do have. Of course, any teacher covering this material will recognize that this leads to additional, potentially confusing complications. For example, a system with two three-variable equations and one one-variable equation could yield a solution to one variable without finding a two-variable equation along the way. There would be no two-variable equation to substitute into to find the second value. This kind of problem solving can be very difficult to teach.

However, this challenge can be approached within the context of the graphic organizer. First reiterate the connection between the number of variables in the equation and whether or not it can be solved. As above, the students will acknowledge that substituting a value for only one variable in either of the equations with three variables will not lead to a solvable equation. Then the students can be guided to an alternative.

> If the three-variable equations have too many variables for you to use in the next step, what kind of equation do you need?
> *Two-variable.*
> In the last example, how did you get the two-variable equations?
> *We combined the three-variable equations.*
> Do we have some three-variable equations we could use in this problem?
> *Yes.*
> Which variable have we already solved for?
> *X.*
> Good. So if we combine the two three-variable equations to eliminate a variable, do we want to eliminate the "x," or should we eliminate another variable so we can substitute the "x" value?
> *Another one.*

Obviously this dialogue leads to a sequence that could permit students to solve these more complicated systems. The dialogue also helps students anticipate what happens if the student eliminates the same variable

that has already been solved, in which case the resulting two-variable equation will not permit a substitution for that variable.

A second issue of generalization is the problem of variable coefficients that do not cancel when added. This challenge requires some additional prior knowledge of common multiples, as indicated earlier in the article. With this prior knowledge, students can be guided to find a way to make initial coefficients equal and then make them opposites, or to make them opposites in a single step, as the students become more sophisticated.

A third direction of generalization involves stepping out of the bounds of systems of three variables. After working with the graphic organizer described here, students can often readily answer the question, "Where would you put an equation with four variables?" They can spontaneously create an organizer of their own with four columns. Other students may require more guidance, but they still seem to have an advantage in the classroom over students who have not worked with the organizer.

Still other types of generalization may not be amenable to this approach. For example, while this organizer is helpful in guiding students through the process of solving a given system of equations, it provides no support for students faced with a word problem or other application where the equations have to be determined before the system can be solved. Further, this organizer is of limited use in trying to generalize to solving nonlinear systems of equations. Nonlinear systems may not be readily solved using linear combinations of equations and may require some other approach, such as substitution. These nonlinear problems may require not only different organizers, but also guidance for students on how to decide whether to try a linear combination approach or a substitution approach.

IMPLICATIONS FOR PRACTITIONERS AND RESEARCHERS

Graphic organizers can have broad applications in upper-level mathematics instruction. They can be used to facilitate instruction of complex procedures, such as the solving of systems of equations, as described in this article, and they can also be used to enhance understanding of mathematics concepts, such as the meaning of negative integer exponents. As such, graphic organizers fill a need for teaching students with strong spatial and nonverbal reasoning skills but relatively poor language skills.

However, the effectiveness of applying graphic organizers to these higher-level mathematics elements has been demonstrated only informally in classrooms. Systematic, controlled research is needed to substantiate these impressions. Such research should determine whether graphic organizers improve mathematics achievement. As a first step, data are currently being

collected for one such study using the organizer for solving systems of linear equations described here with secondary students who have been identified as having learning disabilities and/or attention problems. The goal of this study is to show that augmenting direct strategy instruction by including the graphic organizer will improve performance on several achievement measures, particularly those that emphasize understanding the concepts that justify the procedures for solving systems of linear equations. Anecdotal evidence from the study supports previous classroom experience that the students find the organizer helpful for solving these systems of equations. In fact, in some cases participants have specifically requested permission to use the organizer.

Assuming the effectiveness of the graphic organizer is demonstrated, further research should explore whether the degree of improvement covaries with the nonverbal reasoning and/or spatial abilities of the students participating. If so, this would provide some evidence for an intervention that is differentially effective for some students over others, an important issue in selecting approaches for helping students with learning difficulties. Further research could investigate the effectiveness of graphic organizers applied to other higher-level mathematics content, as well as pursuing efforts to train teachers to use graphic organizers and generate their own graphic organizers. We hope this work will help provide more students with the opportunity to explore these secondary mathematics topics.

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